

THE GEOMETRY OF STRONG KOSZUL ALGEBRAS

EDWARD L. GREEN

ABSTRACT. Koszul algebras with quadratic Gröbner bases, called strong Koszul algebras, are studied. We introduce affine algebraic varieties whose points are in one-to-one correspondence with certain strong Koszul algebras and we investigate the connection between the varieties and the algebras.

1. INTRODUCTION

The connection between affine algebraic varieties and commutative rings, especially quotients of commutative polynomial rings over a field, is well established. In this paper, we introduce a new connection between affine algebraic varieties and a class of Koszul algebra which are not necessarily commutative. The varieties we consider have the property that the points are in one-to-one correspondence with certain Koszul algebras. Given one of these varieties, the Koszul algebras corresponding to the points are shown to have a number of features in common.

To describe our results more precisely, let K be a field, Q a finite quiver, and let KQ denote the path algebra. An element $x \in KQ$ is called *quadratic* if x is a K -linear combination of paths of length 2. We say a (two-sided) ideal I in KQ is a *quadratic ideal* if I can be generated by quadratic elements. One of many equivalent definitions of a *Koszul algebra* is the following: If J is the ideal in KQ generated by the arrows of Q and I is a quadratic ideal in KQ , then KQ/I is a *Koszul algebra* if the Ext-algebra, $\bigoplus_{n \geq 0} \text{Ext}_{KQ/I}^n(KQ/J, KQ/J)$, can be generated in degrees 0 and 1. Although this a very special class of algebras, Koszul algebras occur in many different settings, for example, see [3, 4, 6] and their references.

In this paper we study a class of Koszul algebras which we call *strong Koszul algebras*. To define this class, fix an admissible order \succ on the paths in Q . The formal definition of an admissible order can be found in the beginning of Section 2. Such an order is needed for KQ to have a Gröbner basis theory. We provide a brief overview of the Gröbner basis theory needed for the paper in Section 2. An algebra $\Lambda = KQ/I$ is a *strong Koszul algebra (with respect to I and \succ)*, if I is a quadratic ideal in KQ and I has a Gröbner basis consisting of quadratic elements. That a strong Koszul algebra is in fact a Koszul algebra is proved in [7]. Note that not all Koszul algebras are strong; for example, Sklyanin algebras [13] are Koszul algebras that are not strong. One important type of a strong Koszul algebra is of the form KQ/I^* , where I^* is an ideal that can be generated by a set \mathcal{T} of paths of length 2 in Q . That KQ/I^* is a strong Koszul algebra follows from the fact that \mathcal{T} is a Gröbner basis of I^* with respect to any admissible order [7].

For each set \mathcal{T} of paths of length 2 in KQ , we define an affine algebraic variety $\text{GrAlg}(\mathcal{T})$, such that the points of $\text{GrAlg}(\mathcal{T})$ are in one-to-one correspondence with a particular set of strong Koszul algebras; see Theorem 3.4 and Theorem 3.5. We view this correspondence as an identification and show that each strong Koszul algebra corresponds to a point in one of these varieties; see Corollary 3.3. In each variety $\text{GrAlg}(\mathcal{T})$, there is a distinguished algebra, KQ/I^* , where I^* is generated by \mathcal{T} . All other algebras KQ/I in $\text{GrAlg}(\mathcal{T})$ have the property that I cannot be generated by paths.

The class of strong Koszul algebras includes preprojective algebras whose underlying graph is connected and not a tree [6], straightening closed algebras generated by minors [7], and algebras of the form $KQ/\langle \mathcal{T} \rangle$ where \mathcal{T} is a set of paths of length 2 in KQ and $\langle \mathcal{T} \rangle$ denotes the ideal generated by \mathcal{T} .

The algebras lying in one variety have a number of properties in common. Suppose that $\Lambda = KQ/I$ and $\Lambda' = KQ/I'$ are two strong Koszul algebras in $\text{GrAlg}(\mathcal{T})$. Then we prove the following results. Let $\Lambda^* = KQ/\langle \mathcal{T} \rangle$.

- (1) $\dim_K(\Lambda) = \dim_K(\Lambda') = \dim_K(\Lambda^*)$; see Theorem 4.4.
- (2) Assuming Λ^* is finite dimensional, then $\text{gl. dim}(\Lambda) = \text{gl. dim}(\Lambda') = \text{gl. dim}(\Lambda^*)$; see Corollary 4.7. Note that [5] provides a fast algorithm for computing the global dimension of Λ^* .
- (3) The Betti numbers in the minimal projective resolutions of one dimensional simple modules for all three algebras are the same; see Theorem 4.6.
- (4) The Cartan matrices of Λ , Λ' , and Λ^* are the same; see Corollary 4.5.
- (5) If Λ^* is quasi-hereditary, then so are Λ and Λ' [10]. Furthermore a method for determining if Λ^* is quasi-hereditary is given in [10].

Section 5 is devoted to examples. In particular, the varieties that include the commutative polynomial rings of dimension 2 and 3 are investigated; see Examples 5.2 and 5.4. These examples show that in certain cases, the algebras in the variety $\text{GrAlg}(\mathcal{T})$ are Koszul Artin-Schelter regular algebras, which have played a fundamental role in noncommutative geometry; see, for example, [11, 12] and their references.

Section 6 shows how to restrict the varieties to subvarieties that can be more tractable than the full variety $\text{GrAlg}(\mathcal{T})$. Section 7 shows that if Λ is a strong Koszul algebra, then so are the opposite algebra Λ^{op} and the enveloping algebra, $\Lambda \otimes_K \Lambda^{op}$. The paper ends with some remarks and questions.

2. STRONG KOSZUL ALGEBRAS

To define a strong Koszul algebra we will need to briefly review (graded) Gröbner basis theory. For details we refer the reader to [8]. We fix a field K and a finite quiver Q . The set \mathcal{B} of finite (directed) paths forms a K -basis of the path algebra KQ . We positively \mathbb{Z} -grade $KQ = KQ_0 + KQ_1 + \cdots$ by defining KQ_n to be the K -span of paths in \mathcal{B} of length n . This is called the *length grading* on KQ .

For a Gröbner basis theory we need a special type of order on \mathcal{B} . We say a well-order \succ on \mathcal{B} is *admissible* if, for all p, q, r, s, t in \mathcal{B} ,

- (1) if $p \succ q$, then $pr \succ qr$ if both pr and qr are nonzero,

- (2) if $p \succ q$, then $sp \succ sq$ if both sp and sq are nonzero, and
- (3) if $p = rqt$, then $p \succeq q$.

We fix an admissible order \succ on \mathcal{B} . Since we are interested in graded Koszul algebras where the grading is induced from the length grading of KQ , we add the requirement that if $p, q \in \mathcal{B}$ and $\ell(p) > \ell(q)$ then $p \succ q$, where $\ell(p)$ denotes the path length of p . We call such an admissible order a *length admissible order*. Fix a length admissible order \succ . Note that we give an example of such an order in the beginning of Section 5.

In general, \mathcal{B} will be an infinite set. We make the convention that if $x \in KQ$ and we write $x = \sum_{p \in \mathcal{B}} \alpha_p p$ with $\alpha_p \in K$, then all but a finite number of α_p equal 0. If $x = \sum_{p \in \mathcal{B}} \alpha_p p$ is a nonzero element of KQ , then $\text{tip}(x) = p$ if $\alpha_p \neq 0$ and $p \succeq q$, for all q with $\alpha_q \neq 0$. If $X \subseteq KQ$, then

$$\text{tip}(X) = \{\text{tip}(x) \mid x \in X \text{ and } x \neq 0\}.$$

We say a nonzero element $x \in KQ$ is *uniform* if there exist vertices v and w such that $x = vxw$. Paths are always uniform, and, if Q has one vertex and n loops, then KQ is isomorphic to the free algebra on n noncommuting variables and that every nonzero element of KQ is uniform.

If I is an ideal in KQ , then we say that I is a *graded ideal* if $I = \sum_{n \geq 0} I \cap KQ_n$. Equivalently, I can be generated by (length) homogeneous elements. If I is a graded ideal in KQ and $\Lambda = KQ/I$, then Λ has positive \mathbb{Z} -grading induced from the length grading on KQ , which we call the *induced length grading*.

Definition 2.1. Let I be a graded ideal in KQ and \mathcal{G} a set of length homogeneous uniform elements in I . Then \mathcal{G} is a *(graded) Gröbner basis* of I (with respect to \succ) if

$$\langle \text{tip}(\mathcal{G}) \rangle = \langle \text{tip}(I) \rangle.$$

Definition 2.2. Let $\Lambda = KQ/I$. We say that Λ is a *strong Koszul algebra* (with respect to I and \succ) if I has a Gröbner basis with respect to \succ consisting of quadratic elements.

Theorem 2.3. [7] *A strong Koszul algebra is a Koszul algebra.*

The converse is false in general, for example, the Sklyanin algebras [13].

For the remainder of this section we look more closely at the strong Koszul algebras. If X is a subset of KQ , then we define

$$\text{nontip}(X) = \mathcal{B} \setminus \text{tip}(X)$$

We have the following result whose proof is left to the reader.

Proposition 2.4. . *Let \mathcal{T} be a set of paths in KQ such that if $t, t' \in \mathcal{T}$ and $t \neq t'$ then t is not a subpath of t' . Let $I = \langle \mathcal{T} \rangle$.*

- (1) $n \in \text{nontip}(I)$ if and only if no path in \mathcal{T} is a subpath of n .
- (2) $t = a_1 a_2 \cdots a_n$ with $a_i \in Q_1$ is in \mathcal{T} if and only if $t \notin \text{nontip}(I)$ but $a_2 a_3 \cdots a_n$ and $a_1 a_2 \cdots a_{n-1}$ are in $\text{nontip}(I)$.

□

Our next result is fundamental and is slightly more general than the result found in [8]. Let \mathcal{S} denote the subalgebra of KQ generated by the vertices of Q . Note that \mathcal{S} is a semisimple K -algebra. If \mathcal{X} is a set of paths in Q , $\text{Span}_K(\mathcal{X})$ is an \mathcal{S} -bimodule as follows: if $\sum_{x \in \mathcal{X}} \alpha_x x \in \text{Span}_K(\mathcal{X})$ and $v, w \in Q_0$, then $v(\sum_{x \in \mathcal{X}} \alpha_x x)w = \sum_{x \in \mathcal{X}} \alpha_x (v x w)$. The proof found in [8] can easily be adjusted from K -vector spaces to \mathcal{S} -bimodules.

Lemma 2.5. (Fundamental Lemma) *If I is an ideal in KQ , then*

$$KQ = I \oplus \text{Span}_K(\text{nontip}(I)),$$

as \mathcal{S} -bimodules.

We say an ideal I in KQ is a *monomial* ideal if I can be generated by paths. Note that a monomial ideal is a graded ideal. The proof of the following well-known result is left to the reader.

Proposition 2.6. *Let L be a monomial ideal in KQ . Then*

- (1) *an element $x = \sum_{p \in \mathcal{B}} \alpha_p p$ with $\alpha_p \in K$ is in L if and only if, for each $\alpha_p \neq 0$, $p \in L$, and*
- (2) *there is a unique minimal set of paths that generate L .*

□

We apply the second part of the above proposition and the Fundamental Lemma as follows. Let I be a graded ideal in KQ . Then $\langle \text{tip}(I) \rangle$, the two-sided ideal generated by $\text{tip}(I)$, is a monomial ideal. Hence there is a unique minimal subset, \mathcal{T} , of $\text{tip}(I)$, that generates $\langle \text{tip}(I) \rangle$. By the Fundamental Lemma, for each $t \in \mathcal{T}$, there is a unique $g_t \in I$ and a unique $n(t) \in \text{Span}_K(\text{nontip}(I))$ such that $t = g_t + n(t)$. In particular, for each $t \in \mathcal{T}$, $t - n_t \in I$.

Proposition 2.7. *The set $\mathcal{G} = \{g_t \mid t \in \mathcal{T}\}$ is a graded Gröbner basis for I .*

Proof. Each $g_t \in I$ implies that $\text{tip}(g_t) \in \text{tip}(I)$. Since $n(t) \in \text{Span}_K(\text{nontip}(I))$, we conclude that, for each $t \in \mathcal{T}$, $\text{tip}(g_t) = t$. Next we show that \mathcal{G} consists of uniform length homogeneous elements. Letting $t \in \mathcal{T}$ be a path of length m , writing $t = g_t + n(t)$ we see that, in degree m , $(g_t)_m \in I$ and $n(t)_m$ remains in $\text{Span}_K(\text{nontip}(I))$. We have that $t = (g_t)_m + n(t)_m$ and, by unicity, each g_t is a length homogeneous element. The proof that each g_t is uniform is similar. Since \mathcal{T} generates $\langle \text{tip}(I) \rangle$, $\mathcal{T} = \text{tip}(\mathcal{G})$, the elements of \mathcal{G} are uniform, length homogenous, and hence we are done. □

Definition 2.8. Given a graded ideal I in KQ and \mathcal{G} as constructed above, we call \mathcal{G} the *reduced* Gröbner basis for I (with respect to \succ).

Returning to strong Koszul algebras, if an ideal has a Gröbner basis \mathcal{H} of uniform quadratic elements then $\text{tip}(\mathcal{H})$ consists of paths of length 2, and hence the reduced Gröbner basis consists of quadratic uniform elements. Thus,

Proposition 2.9. *We have that $\Lambda = KQ/I$ is a strong Koszul algebra if and only if the reduced Gröbner basis consists of quadratic uniform elements.*

3. THE VARIETY $\text{GrAlg}(\mathcal{T})$

In this section, if \mathcal{T} is a set of paths of length 2, we define an affine variety whose points are in one-to-one correspondence to the strong Koszul algebras $\Lambda = KQ/I$ (with respect to I and \succ), having the property that $\langle \text{tip}(I) \rangle$ is generated by \mathcal{T} . Referring to Example 5.1 while reading this section should be helpful.

Fix \mathcal{T} to be a set of paths of length 2. Set $\mathcal{N} = \mathcal{B} \setminus \text{tip}(\langle \mathcal{T} \rangle)$. Recall that if I is an ideal such that $\langle \text{tip}(I) \rangle = \langle \mathcal{T} \rangle$, then $\mathcal{N} = \text{nontip}(I)$. It is important to note that \mathcal{N} is only dependent on \mathcal{T} and not on I .

We begin by defining the affine space in which our variety lives. For this we need the following definitions. We say two elements x and y of KQ are *parallel* if there are vertices v and w such that $vwx = x$ and $vwy = y$. In particular, if x and y are parallel then both x and y are uniform. If x and y are parallel, we write $x \parallel y$. Note that if $x = \sum_{p \in \mathcal{B}} \alpha_p p \in KQ$, then x is uniform if and only if $p \parallel q$, for all $p, q \in \mathcal{B}$ with α_p and α_q nonzero.

Let \mathcal{N}_2 be the set of paths of length 2 in $\mathcal{N} = \mathcal{B} \setminus \text{tip}(\langle \mathcal{T} \rangle)$ and, for $t \in \mathcal{T}$, define

$$\mathcal{N}_2(t) = \{n \in \mathcal{N}_2 \mid t \succ n \text{ and } n \parallel t\}.$$

We now can define the affine space in which our variety lives. If S is a set, then $|S|$ denotes the cardinality of S . Let $D = \sum_{t \in \mathcal{T}} |\mathcal{N}_2(t)|$. We let $\mathcal{A} = K^D$, viewed as a D -dimensional affine space. If $X \in \mathcal{A}$, then we write X as tuple with indices in the disjoint union of the $\mathcal{N}_2(t)$'s; that is, we write $X = (x_{t,n})$, where $t \in \mathcal{T}$, $n \in \mathcal{N}_2(t)$, and $x_{t,n} \in K$.

For each $X = (x_{t,n}) \in \mathcal{A}$, let

$$\mathcal{G}(X) = \{g_t \in KQ \mid g_t = t - \sum_{n \in \mathcal{N}_2(t)} x_{t,n} n\}.$$

We now define the subset of \mathcal{A} of interest.

Definition 3.1. Given a set \mathcal{T} of paths of length 2, define

$$\text{GrAlg}(\mathcal{T}) = \{X \in \mathcal{A} \mid KQ/\langle \mathcal{G}(X) \rangle \text{ is a strong Koszul algebra (with respect to } \langle \mathcal{G}(X) \rangle \text{ and } \succ)\}$$

The remainder of this section is devoted to showing that $\text{GrAlg}(\mathcal{T})$ is an affine variety in \mathcal{A} whose points are in one-to-one correspondence with the elements of

$$\begin{aligned} \mathcal{U} = & \text{the set of algebras } KQ/I \text{ that are strong Koszul algebras} \\ & \text{(with respect to } I \text{ and } \succ) \text{ such that } \langle \mathcal{T} \rangle = \langle \text{tip}(I) \rangle. \end{aligned}$$

First we will show the one-to-one correspondence. We begin with a preparatory result.

Proposition 3.2. *If $KQ/I \in \mathcal{U}$ then there exists $X \in \mathcal{A}$ such that the reduced Gröbner basis of I is $\mathcal{G}(X)$ for some X . Moreover, X is unique.*

Proof. Suppose that $KQ/I \in \mathcal{U}$. Since $\langle \mathcal{T} \rangle = \langle \text{tip}(I) \rangle$ and \mathcal{T} are paths of length 2, \mathcal{T} must be the unique minimal generating set of the monomial ideal $\langle \text{tip}(I) \rangle$. It now follows from our discussion of the reduced Gröbner basis that there is some $X \in \mathcal{A}$ such that $I = \langle \mathcal{G}(X) \rangle$. Uniqueness follows from the uniqueness of the reduced Gröbner basis. \square

Corollary 3.3. *If $\Lambda = KQ/I$ is a strong Koszul algebra (with respect to I and \succ), then $\Lambda \in \text{GrAlg}(\mathcal{T})$ where \mathcal{T} is the minimal set of generators of $\langle \text{tip}(I) \rangle$.*

Proof. The reduced Gröbner basis \mathcal{G} of I with respect to \succ is composed of uniform quadratic elements. Let $\mathcal{T} = \text{tip}(\mathcal{G})$. It is immediate that \mathcal{T} is the minimal set of paths that generate $\langle \text{tip}(I) \rangle$. It is now clear that $\Lambda \in \text{GrAlg}(\mathcal{T})$. \square

We now state the correspondence theorem.

Theorem 3.4. *Let \mathcal{T} be a set of paths of length 2. There is a one-to-one correspondence between the points of $\text{GrAlg}(\mathcal{T})$ and the algebras KQ/I that are strong Koszul algebras (with respect to I and \succ) such that $\langle \mathcal{T} \rangle = \langle \text{tip}(I) \rangle$.*

Proof. Define $\varphi: \text{GrAlg}(\mathcal{T}) \rightarrow \mathcal{U}$ by $\varphi(X) = KQ/\langle \mathcal{G}(X) \rangle$. The map φ is well-defined. We see that φ is injective, since if $X = (x_{t,n}), X' = (x'_{t,n}) \in \text{GrAlg}(\mathcal{T})$ with $X \neq X'$, the reduced Gröbner bases of $\varphi(X)$ and $\varphi(X')$ differ. But the reduced Gröbner basis of an ideal is unique, and φ being injective follows.

To see that φ is onto, let $KQ/I \in \mathcal{U}$. We are assuming that $\langle \text{tip}(I) \rangle = \langle \mathcal{T} \rangle$. Since every path in \mathcal{T} is a path of length 2 and $\langle \text{tip}(I) \rangle = \langle \mathcal{T} \rangle$, we see that \mathcal{T} is the (unique) minimal set of paths that generate $\langle \text{tip}(I) \rangle$. By the construction of the reduced Gröbner basis for I with respect to \succ found in Section 2, the reduced Gröbner basis for I with respect to \succ is $\{g_t \mid t \in \mathcal{T} \text{ and } g_t = t - \sum_{n \in \mathcal{N}_2(\mathcal{T})} x_{t,n}n\}$ for some $x_{t,n} \in K$. Thus, $KQ/I = \varphi((x_{t,n}))$, and we are done. \square

We now show the somewhat surprising result that $\text{GrAlg}(\mathcal{T})$ is an affine algebraic variety in \mathcal{A} .

Theorem 3.5. *Let K be a field, Q a finite quiver, and \mathcal{T} be a set of paths of length 2 in Q . Then $\text{GrAlg}(\mathcal{T})$ is an affine algebraic variety.*

Before proving this result, we will need some preliminary work.

We introduce a “polynomial ring over a path algebra”. Let \mathbf{y} be a set of D variables with $\{y_{t,n}\} = \mathbf{y}$ where $t \in \mathcal{T}$ and $n \in \mathcal{N}_2(t)$. Consider the ring $R = KQ[\mathbf{y}]$, consisting of finite sums of the form $\sum_{p \in \mathcal{B}} f_p(\mathbf{y})p$, where $f_p(\mathbf{y})$ is a polynomial in the commutative polynomial ring $K[\mathbf{y}]$. The variables $y_{t,n}$ commute with elements of the path algebra KQ (and each other). Note that $K[\mathbf{y}]$ is the coordinate ring of the affine space \mathcal{A} . Given an element $\sum_{p \in \mathcal{B}} f_p(\mathbf{y})p \in R$, we call the polynomial $f_p(\mathbf{y})$ the ‘coefficient’ of p

We are interested in a particular set of elements in R , namely

$$\mathcal{H} = \{h_t \in R \mid h_t = t - \sum_{n \in \mathcal{N}_2(t)} y_{t,n} n\}.$$

If $F = \sum_{p \in \mathcal{B}} f_p(\mathbf{y})p$ is an element of R , then we say $F' \in R$ is a *simple reduction* of F by \mathcal{H} , written $F \rightarrow_{\mathcal{H}} F'$, if there is some $p \in \mathcal{B}$ and $t \in \mathcal{T}$ such that

- (1) $p = qtr$ for some paths q and r ,
- (2) $f_p(\mathbf{y}) \neq 0$, and
- (3) $F' = F - f_p(\mathbf{y})p + (f_p(\mathbf{y})(q(\sum_{n \in \mathcal{N}_2(t)} y_{n,t} n)r))$.

The effect of a simple reduction is the following. Suppose $f_p(\mathbf{y})p$ occurs in F with $p = qtr$ and it is the term we work with. Then the term $f_p(\mathbf{y})p$ is replaced with the sum of terms $(f_p(\mathbf{y})y_{n,t})qtr$, for $n \in \mathcal{N}_2(t)$. Note that $p \succ qtr$ for each $n \in \mathcal{N}_2(t)$. Thus, for each $n \in \mathcal{N}_2(t)$, $f_p(\mathbf{y})y_{n,t}$ is added to $f_{qtr}(\mathbf{y})$ as the 'coefficient' in front of $p = qtr$ and $f_p(\mathbf{y})p$ is removed. All other terms in F are unchanged.

We say F^* is a *complete reduction* of F by \mathcal{H} , written $F \Rightarrow_{\mathcal{H}} F^*$, if, for some m , there is a sequence $F_1 = F, F_2, \dots, F_m = F^*$ such that for each $i = 1, \dots, m-1$, $F_i \rightarrow_{\mathcal{H}} F_{i+1}$ is a simple reduction, and F^* has no simple reduction. Note that F^* having no simple reduction is equivalent to saying that all the paths p in F^* having nonzero coefficient in $K[\mathbf{y}]$ are in \mathcal{N} ; that is, for all $t \in \mathcal{T}$, t is not a subpath of any p occurring in F^* . Since \succ is a well-order on \mathcal{B} , every $F \in R$ will have a complete reduction.

We now prove Theorem 3.5. Recall that $\mathcal{H} = \{h_t \in R \mid h_t = t - \sum_{n \in \mathcal{N}_2(t)} y_{t,n} n\}$. For each pair t and t' of elements of \mathcal{T} such that $t = ab$ and $t' = bc$, where a, b , and c are arrows in Q , form the *overlap relation*

$$Ov(t, t') = h_t \cdot c - a \cdot h_{t'}.$$

Note that $t = t' = a^2$ is allowed. Since each h_t is a $K[\mathbf{y}]$ -combination of paths of length 2, each overlap relation is a $K[\mathbf{y}]$ -combination of paths of length 3. We note that if we have a $K[\mathbf{y}]$ -combination of paths of length 3, then a simple reduction is again a $K[\mathbf{y}]$ -combination of paths of length 3. It follows that a complete reduction of a $K[\mathbf{y}]$ -combination of paths of length 3 is again a $K[\mathbf{y}]$ -combination of paths of length 3.

Let \mathcal{N}_3 denote the set of paths in \mathcal{N} of length 3. For each $Ov(t, t')$, let

$$F_{t,t'}^* = \sum_{\hat{n} \in \mathcal{N}_3} f_{t,t',\hat{n}}^*(\mathbf{y})\hat{n},$$

with $f_{t,t',\hat{n}}^*(\mathbf{y}) \in K[\mathbf{y}]$, be a complete reduction of $Ov(t, t')$ by \mathcal{H} . Thus, for each $t = ab$, $t' = bc$ and each $\hat{n} \in \mathcal{N}_3$, we obtain polynomials in commutative polynomial ring $K[\mathbf{y}]$, namely, the coefficient $f_{t,t',\hat{n}}^*(\mathbf{y})$ of \hat{n} in $F_{t,t'}^*$.

We claim that $\text{GrAlg}(\mathcal{T})$ is the zero set of

$$\mathcal{I} = \{f_{t,t',\hat{n}}^*(\mathbf{y}) \mid \hat{n} \in \mathcal{N}_3, t, t' \in \mathcal{T} \text{ with } t = ab \text{ and } t' = bc, \text{ for some arrows } a, b, c\}.$$

If, in the definitions of overlap relation, simple reduction, and complete reduction, instead of variables $y_{t,n}$ we use elements $x_{t,n}$ in K , we would have the definitions of overlap relation, simple reduction, and complete reduction for elements of the path algebra KQ . The noncommutative version of Buchberger's Theorem [8, 2], applied to our setup, says that if $\mathcal{G} = \{g_t \mid t \in \mathcal{T}\}$ is a uniform set of quadratic elements in KQ

such that $\text{tip}(g_t) = t$, then \mathcal{G} is a Gröbner basis for $\langle \mathcal{G} \rangle$ if and only if all overlap relations completely reduce to 0.

Suppose that $X = (x_{t,n}) \in \text{GrAlg}(\mathcal{T})$. We show that X is in the zero set of \mathcal{I} . We note that $\mathcal{G}(X) = \{g_t = t - \sum_{n \in \mathcal{N}_2(t)} x_{t,n} n \mid t \in \mathcal{T}\}$ is just \mathcal{H} evaluated at X . Since $X \in \text{GrAlg}(\mathcal{T})$, $\mathcal{G}(X)$ is the reduced Gröbner basis for $\langle \mathcal{G}(X) \rangle$ and hence all overlap relations of $\mathcal{G}(X)$ reduce to 0. Thus each $f_{t,t',\hat{n}}^*(X) = 0$; in particular X is in the zero set of \mathcal{I} .

Conversely, if X is in the zero set of \mathcal{I} , each $f_{t,t',\hat{n}}^*(X) = 0$. Hence every overlap relation of $\mathcal{G}(X)$ completely reduces to 0, and we conclude that $\mathcal{G}(X)$ is a Gröbner basis of the ideal $\langle \mathcal{G}(X) \rangle$. Since $\text{tip}(\mathcal{G}(X)) = \mathcal{T}$ we see that $X \in \text{GrAlg}(\mathcal{T})$. This completes the proof. \square

4. PROPERTIES OF $\text{GrAlg}(\mathcal{T})$

We begin with a general definition.

Definition 4.1. Given $\Lambda = KQ/I$, an arbitrary algebra, $KQ/\langle \text{tip}(I) \rangle$ is called the *associated monomial algebra* of Λ and denoted Λ_{Mon} . We also define I_{Mon} to be $\langle \text{tip}(I) \rangle$.

Note that given an ideal I , $\text{tip}(I)$ is dependent on the choice of the admissible order \succ and that, in this paper, \succ is fixed and has the property that, if $p, q \in \mathcal{B}$ and the length of p is greater than the length of q , then $p \succ q$.

The next result provides an alternative definition of $\text{GrAlg}(\mathcal{T})$. Recall that if $\mathbf{x} = (x_{t,n}) \in \mathcal{A} = K^D$, then $\mathcal{G}(\mathbf{x}) = \{g_t = t - \sum_{n \in \mathcal{N}_2(t)} x_{t,n} n \mid t \in \mathcal{T}\}$.

Proposition 4.2. Let \mathcal{T} be a set of paths of length 2 in a quiver Q . Let $\mathbf{0} = (0, 0, \dots, 0) \in \mathcal{A}$. The following statements hold:

- (1) $\mathcal{G}(\mathbf{0}) = \mathcal{T}$.
- (2) The element $\mathbf{0} \in \mathcal{A}$ is in $\text{GrAlg}(\mathcal{T})$ and corresponds to the strong Koszul algebra $KQ/\langle \mathcal{T} \rangle$ (with respect to $\langle \mathcal{T} \rangle$ and \succ).
- (3) Let $\Lambda = KQ/I$ be a length graded algebra. Then Λ is a strong Koszul algebra (with respect to I and \succ) corresponding to a point in $\text{GrAlg}(\mathcal{T})$ if and only if $I_{\text{Mon}} = \langle \mathcal{T} \rangle$.
- (4) If I is an ideal in KQ generated by length homogeneous elements, then KQ/I corresponds to an element in the zero set of \mathcal{I} if and only if $(KQ/I)_{\text{Mon}} = KQ/(I_{\text{Mon}}) = KQ/\langle \mathcal{T} \rangle$.
- (5) There is exactly one algebra with quadratic monomial Gröbner basis that corresponds to a point in $\text{GrAlg}(\mathcal{T})$, namely, $KQ/\langle \mathcal{T} \rangle$.

\square

The proof is straightforward and left to the reader. As a consequence, we have the following corollary.

Corollary 4.3. *Let $\Lambda = KQ/I$ be a K -algebra with length grading induced from the length grading of KQ . The following statements are equivalent:*

- (1) Λ corresponds to an element of $\text{GrAlg}(\mathcal{T})$.
- (2) $\Lambda_{\text{Mon}} = KQ/\langle \mathcal{T} \rangle$.
- (3) $I_{\text{Mon}} = \langle \mathcal{T} \rangle$.

The next result shows that two algebras in a variety have the same K -bases of paths under the splitting of the canonical surjection $\pi: KQ \rightarrow KQ/I$ given by the Fundamental Lemma. More precisely, let $\sigma: KQ/I \rightarrow KQ$ be defined by $\sigma(\pi(x)) = n_x$ where $x = i_x + n_x$ with $i_x \in I$ and $n_x \in \text{Span}_K(\text{nontip}(I))$. The map σ is well-defined by the Fundamental Lemma, and $\pi\sigma = 1_{KQ/I}$. We identify $\Lambda = KQ/I$ with $\text{Span}_K(\text{nontip}(I))$. If $\Lambda \in \text{GrAlg}(\mathcal{T})$, then $\text{nontip}(I) = \mathcal{B} \setminus \text{tip}(I)$. Now $\langle \text{tip}(I) \rangle = \langle \mathcal{T} \rangle$ and hence $\text{nontip}(I) = \{n \in \mathcal{B} \mid n \text{ has no subpath in } \mathcal{T}\}$ by Proposition 2.4. Thus, every algebra in $\text{GrAlg}(\mathcal{T})$ has as K -basis $\{n \in \mathcal{B} \mid n \text{ has no subpath in } \mathcal{T}\}$. Of course, multiplication of elements of the basis differs for different algebras.

The converse holds. More precisely, if $\Lambda = KQ/I$, where I is generated by uniform quadratic elements and $\text{nontip}(I) = \{n \in \mathcal{B} \mid n \text{ has no subpath in } \mathcal{T}\}$, then Λ is in $\text{GrAlg}(\mathcal{T})$. To see this, since $\mathcal{B} = \text{tip}(I) \oplus \text{nontip}(I)$, it follows that $\text{tip}(I) = \{p \in \mathcal{B} \mid \text{there is some } t \in \mathcal{T} \text{ such that } t \text{ is a subpath of } p\}$. From this description it follows that $\langle \text{tip}(I) \rangle = \langle \mathcal{T} \rangle$, and hence $\Lambda \in \text{GrAlg}(\mathcal{T})$.

We summarize the above discussion in the next result, which provides another description of $\text{GrAlg}(\mathcal{T})$.

Theorem 4.4. *Let \mathcal{T} be a set of paths of length 2 in Q and $\mathcal{N} = \mathcal{B} \setminus \text{tip}(\langle \mathcal{T} \rangle)$. We have that if I is generated by uniform length homogeneous elements, then $\Lambda = KQ/I \in \text{GrAlg}(\mathcal{T})$ if and only if $\mathcal{N} = \text{nontip}(I)$. Moreover, if $\Lambda = KQ/I \in \text{GrAlg}(\mathcal{T})$ then, as a subspace of KQ , Λ has K -basis \mathcal{N} . \square*

The following consequence of the previous theorem describes the Cartan matrix of a strong Koszul algebra and shows that two algebras in the same variety have the same Cartan matrix.

Corollary 4.5. *Let \mathcal{T} and \mathcal{N} be as in Theorem 4.4 and $\Lambda = KQ/I \in \text{GrAlg}(\mathcal{T})$. Suppose that $|\mathcal{N}| < \infty$ and $\{v_1, \dots, v_n\} = Q_0$. Then the Cartan matrix of Λ is the $n \times n$ matrix C where the (i, j) -th entry in C is $|\mathcal{N}_{v_i v_j}|$, the number of paths from i to j in \mathcal{N} .*

Proof. The Cartan matrix is the $n \times n$ matrix with (i, j) -th entries $\dim_K(v_i \Lambda v_j)$. But $\text{Span}_K(\text{nontip}(I))$ is isomorphic to Λ and $\mathcal{N} = \text{nontip}(I)$

\square

The next result shows that algebras in the same variety share some homological properties. Assume I is an ideal in KQ contained in J^2 , where J is the ideal in KQ generated by the arrows of Q . If v is a vertex in Q and $\Lambda = KQ/I$, we let $S_v(\Lambda)$ be the one-dimensional simple Λ -module associated to the vertex v .

Note that if $\Lambda = KQ/I \in \text{GrAlg}(\mathcal{T})$, $I \subseteq J^2$ since I has a Gröbner bases consisting of quadratic elements.

If Λ is a ring and M is a Λ -module, we let $\text{pd}_\Lambda(M)$ and $\text{id}_\Lambda(M)$ denote the projective and injective dimensions of M respectively.

Theorem 4.6. *Let \mathcal{T} be a set of paths of length 2 in Q and $\Lambda \in \text{GrAlg}(\mathcal{T})$. Suppose that v and w are vertices in Q . Then for $n \geq 0$,*

$$\dim_K(\text{Ext}_\Lambda^n(S_v(\Lambda), S_w(\Lambda))) = \dim_K(\text{Ext}_{\Lambda^*}^n(S_v(\Lambda^*), S_w(\Lambda^*)),$$

where $\Lambda^* = KQ/\langle \mathcal{T} \rangle$. In particular, if $\mathcal{N} = \mathcal{B} \setminus \text{tip}(\langle \mathcal{T} \rangle)$, then

$$\text{pd}_\Lambda(S_v(\Lambda)) = \text{pd}_{\Lambda^*}(S_v(\Lambda^*)) \text{ and } \text{id}_\Lambda(S_v(\Lambda)) = \text{id}_{\Lambda^*}(S_v(\Lambda^*)).$$

.

Proof. Although the proof of this result is implicit in [1], we sketch a proof employing the ideas in [9]. In [9], a projective Λ -resolution is constructed inductively from subsets $F^n = \{f_i^n\}_{i=1}^{k_i}$ for $n \geq 0$, where $f_i^n \in KQ$ and k_i is finite if the reduced Gröbner basis is finite, which it is in our case. For $S_v(\Lambda)$, we have $F^0 = \{v\}$, F^1 is the set of arrows starting at v , and F^2 are the elements of a reduced Gröbner basis that start at v . The F^n 's are constructed using the F^i 's, $i < n$ and the reduced Gröbner basis.

We describe $\text{tip}(F^n)$, for $n \geq 2$ which can be deduced from the construction of F^n from F^{n-1} . The construction shows that $\text{tip}(F^n) = T^n$, where

$$T^n = \{a_1 a_2 \cdots a_n \mid a_i \in Q_1, v a_1 = a_1, \text{ and } a_i a_{i+1} \in \mathcal{T}, \text{ for } 1 \leq i \leq n-1\}.$$

Note that T^n depends only on \mathcal{T} , and that $|T^n| = |F^n|$. Since the f_i^n are length homogeneous, we see that if $f_i^n \in F^n$ and $\text{tip}(f_i^n) = a_1 \cdots a_n \in T^n$, then f_i^n is length homogeneous of length n . Since each f_i^n is uniform, if w is the end vertex of a_n , then $v f_i^n w = f_i^n$.

From the construction of the $\{f_i^n\}$, each f_i^n is a sum of elements of the form $f_j^{n-1} r_{i,j}$ with $r_{i,j} \in KQ$. By length, the $r_{i,j}$ are linear combinations of arrows. Since the $r_{i,j}$ modulo I are entries in the matrix mapping the n^{th} projective to the $n-1^{\text{st}}$ in the constructed projective resolution of $S_v(\Lambda)$, we conclude that the resolution constructed in [9] is minimal in our case.

Finally, we see that if F^n is the set for resolving $S_v(\Lambda)$, and F^{*n} is the set for resolving $S_v(\Lambda^*)$, then they both have tip set T^n . This finishes the proof since the dimension of $\text{Ext}_\Lambda^n(S_v(\Lambda), S_w(\Lambda))$ equals the number of f_i^n 's such that $v f_i^n w = f_i^n$. \square

We have the following consequence.

Corollary 4.7. *If $\mathcal{N} = \mathcal{B} \setminus \text{tip}(\langle \mathcal{T} \rangle)$ is a finite set and $\Lambda, \Lambda' \in \text{GrAlg}(\mathcal{T})$, then*

$$\text{gl. dim}(\Lambda) = \text{gl. dim}(\Lambda').$$

Proof. Let $\Lambda, \Lambda' \in \text{GrAlg}(\mathcal{T})$. Since $|\mathcal{N}| = \dim_K(\Lambda) = \dim_K(\Lambda')$, $\text{gl. dim}(\Lambda) = \max_{v \in Q_0} \{\text{pd}_\Lambda(S_v(\Lambda))\}$ and $\text{gl. dim}(\Lambda') = \max_{v \in Q_0} \{\text{pd}_{\Lambda'}(S_v(\Lambda'))\}$. The result follows from Theorem 4.6. \square

If $\Lambda^* = KQ/\langle \mathcal{T} \rangle$ is a finite dimensional strong Koszul algebra (with respect to $\langle \mathcal{T} \rangle$ and \succ) and of finite global dimension, then the determinant of the Cartan matrix for every algebra in $\text{GrAlg}(\mathcal{T})$ is 1, since every algebra in $\text{GrAlg}(\mathcal{T})$ is length graded and of finite global dimension and, hence, we may apply [14].

The final property is one that is proved in a more general setting in [10].

Theorem 4.8. [10] *Let \mathcal{T} be set of paths of length 2 in Q and $\mathcal{N} = \mathcal{B} \setminus \text{tip}(\langle \mathcal{T} \rangle)$. Assume that \mathcal{N} is a finite set. If $KQ/\langle \mathcal{T} \rangle$ is a quasi-heredity algebra, then every algebra in $\text{GrAlg}(\mathcal{T})$ is quasi-heredity.*

5. EXAMPLES

We begin by defining a particular length admissible order. Unlike the commutative case, the (left) lexicographic order is not a well-order on \mathcal{B} . In particular, the lexicographic order will have infinite descending chains, contradicting that it must be a well-order.

We describe the *length-left-lexicographic order* which is length admissible. Arbitrarily linearly order the vertices and arrows of Q and require that every vertex is less than every arrow. If $p = a_1 a_2 \cdots a_n$ and $q = b_1 b_2 \cdots b_m$ are paths with the a_i and b_j arrows, then $p \succ q$ if $n > m$ or $n = m$ and there is k , $1 \leq k \leq n$ such that for $1 \leq i \leq k-1$, $a_i = b_i$ and $a_k \succ b_k$.

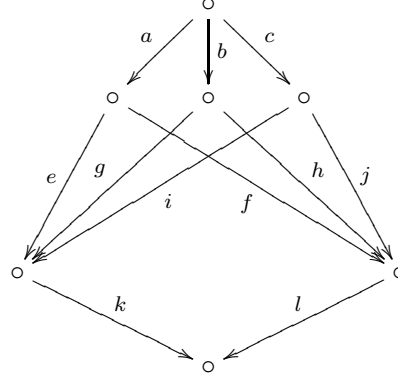
In all the examples in this section we will use the length-left-lexicographic order and it will suffice, in our setting, to simply give the ordering of the arrows. Recall that if \mathcal{T} is a set of paths of length 2 in a quiver Q and $\mathcal{G} = \{g_t \mid t \in \mathcal{T}\}$ where, for $t \in \mathcal{T}$, $g_t = t - \sum_{n \in \mathcal{N}_2(t)} x_{t,n} n$ for $n \in \mathcal{N}_2(t)$, $x_{t,n} \in K$, then \mathcal{G} is the reduced Gröbner basis for the ideal $\langle \mathcal{G} \rangle$ if and only if all overlap relations completely reduce to 0 by \mathcal{G} .

There are two extreme cases given \mathcal{T} and \mathcal{G} as above. The first occurs if there are no overlap relations. In this case, the ideal of the variety $\text{GrAlg } \mathcal{T}$ is (0) and hence $\text{GrAlg } \mathcal{T}$ is all of affine space and there are no restrictions on the choice of the $x_{t,n}$. The second case extreme case occurs if $\mathcal{N}_2(t) = \emptyset$, for all $t \in \mathcal{T}$. For example, this occurs if for each $t \in \mathcal{T}$, if $t \succ t'$ and t' is a path of length 2, then $t' \in \mathcal{T}$. In this case, the affine space \mathcal{A} is dimension 0 and $\text{GrAlg}(\mathcal{T}) = \mathcal{A}$ is a point, namely the monomial algebra $KQ/\langle \mathcal{T} \rangle$.

We now turn to specific examples. The next example is designed to help understand the proof of Theorem 3.5.

Example 5.1. Let Q be the quiver

and let \succ be



defined by $a \succ b \succ \dots \succ l$. Consider $\mathcal{T} = \{af, ae, bg, bh, ek, gk, ik\}$. Then, it follows that $\mathcal{N} = Q_0 \cup \{a, b, c, \dots, k, l, ci, cj, fl, hl, jl, cfk, cjl\}$. Thus we have $\mathcal{N}_2(af) = \{cj\}$, $\mathcal{N}_2(ae) = \{ci\}$, $\mathcal{N}_2(bg) = \{ci\}$, $\mathcal{N}_2(bh) = \{cj\}$, $\mathcal{N}_2(ek) = \{fl\}$, $\mathcal{N}_2(gk) = \{hl\}$, $\mathcal{N}_2(ik) = \{jl\}$.

Thus $\text{GrAlg}(\mathcal{T})$ is a variety in $\mathcal{A} = K^7$. Simplify notation by renaming the variables $y_{t,n}$ where $t \in \mathcal{T}$ and $n \in \mathcal{N}_2(t)$ as follows: $y_{af,cj} = X_1$, $y_{ae,ci} = X_2$, $y_{bg,ci} = X_3$, $y_{bh,cj} = X_4$, $y_{ek,fl} = X_5$, $y_{gk,hl} = X_6$, $y_{ik,jl} = X_7$. We let

$$\mathcal{G} = \{af - X_1cj, ae - X_2cf, bg - X_3ci, \dots, ik - X_7jl\}.$$

There are 2 overlap relations $ov(ae, ek) = -X_2cik + X_5afl$ and $ov(bg, gk) = -X_3cik + X_6bjl$. We completely reduce the first overlap relation and leave the computation of the second to the reader. We see that $-X_2cik$ simply reduces to $-X_7X_2cjl$ by $ik - X_7jl$. Since $cjl \in \mathcal{N}$ it has no simple reductions. Next consider the second term X_5afl . Using $af - X_1cj$, X_5afl simply reduces to X_1X_5cjl and, as we noted, $cjl \in \mathcal{N}$. Thus $ov(ae, ek)$ completely reduces to $-X_2X_7cjl + X_1X_5cjl$. Similarly, $ov(bg, gk) \Rightarrow_{\mathcal{G}} -X_3X_7cjl + X_4X_6cjl$.

Thus, the ideal of the variety is $\mathcal{I} = \langle X_1X_5 - X_2X_7, X_4X_6 - X_3X_7 \rangle$ in the commutative polynomial ring $K[X_1, X_2, \dots, X_7]$. Note that using [5] we see that the $\text{gl.dim}(KQ/\langle \mathcal{T} \rangle) = 3$. Thus by Corollary 4.7 and Theorem 4.4, every algebra in $\text{GrAlg}(\mathcal{T})$ is a strong Koszul algebra of dimension 26 with K -basis \mathcal{N} , and has global dimension 3. \square

We denote the free associative algebra in n variables by $K\{x_1, \dots, x_n\}$. Our next example is very small and simple. In this example, $\text{GrAlg}(\mathcal{T})$ is affine 2-space and there is a punctured line in $\text{GrAlg}(\mathcal{T})$ consisting of the quantum affine planes. Moreover there is another line in $\text{GrAlg}(\mathcal{T})$ on which all the algebras are isomorphic to the monomial algebra $K\{x, y\}/\langle \mathcal{T} \rangle$. This provides an example of distinct points in $\text{GrAlg}(\mathcal{T})$ corresponding to isomorphic algebras.

Example 5.2. Let $R = K\{x, y\}/\langle xy - yx \rangle$ and $y \succ x$. In this example, Q has one vertex and two loops. Then $\mathcal{G} = \{yx - xy\}$ and $\mathcal{T} = \{yx\}$. The paths of length 2 are ordered $y^2 \succ yx \succ xy \succ x^2$. We see that $\mathcal{N} = \{x^i y^j \mid i, j \geq 0\}$. $\mathcal{N}_2(yx) = \{xy, x^2\}$. Thus, $\text{GrAlg}(\{yx\})$ lives in $\mathcal{A} = K^2$. There are no overlap relations and hence $\text{GrAlg}(\{yx\}) = \mathcal{A}$. Thus, every algebra of the form $\Lambda_{(\lambda, \gamma)} = K\{x, y\}/\langle yx - \lambda xy - \gamma x^2 \rangle$ is in $\text{GrAlg}(\{yx\})$, where $(\lambda, \gamma) \in K^2$. Note that in the notation of Section 3, $\lambda = y_{yx, xy}$ and $\gamma = y_{yx, x^2}$.

If $\gamma = 0$ and $\lambda \neq 0, 1$, then $\Lambda_{(\lambda,0)}$ is a quantum affine plane. They all lie on the punctured line $L = \{(\lambda, 0) \mid \lambda \in K \setminus \{0, 1\}\}$ in \mathcal{A} . We also have $\Lambda_{(1,0)} = R$, the commutative polynomial ring in 2 variables. Of course, $\Lambda_{(0,0)}$ is the (noncommutative) monomial algebra $K\{x, y\}/\langle yx \rangle$.

On the other hand, the line determined by $\lambda = 0$ consists of the algebras $K\{x, y\}/\langle yx - \gamma x^2 \rangle$. It is not hard to show these algebras are all isomorphic to each other. In particular, they are all isomorphic to the monomial algebra $K\{x, y\}/\langle yx \rangle$.

If both λ and γ are not 0, then other strong Koszul algebras occur.

Finally, for every $\Lambda \in \text{GrAlg}(\{yx\})$ the minimal projective Λ -resolution of K looks like

$$0 \rightarrow \Lambda \rightarrow \Lambda^2 \rightarrow \Lambda \rightarrow K \rightarrow 0.$$

□

In the next small example, the algebras occurring are finite dimensional.

Example 5.3. Again take Q having one vertex and two loops, x and y . Again we order $y \succ x$. Let $\mathcal{T} = \{x^2, y^2, yx\}$. Then $\mathcal{N} = \{1, x, y, xy\}$. Hence $\mathcal{N}_2(x^2) = \emptyset$, $\mathcal{N}_2(y^2) = \{xy\} = \mathcal{N}_2(yx)$. Thus \mathcal{A} is two space. Now let U and V be variables. We wish to find the ideal \mathcal{I} of the variety $\text{GrAlg}(\mathcal{T})$. We have $\mathcal{G} = \{x^2, y^2 - Uxy, yx - Vxy\}$. Note that in the notation of Section 3, $U = y_{y^2, xy}$ and $V = y_{yx, xy}$. To find the polynomials in \mathcal{I} , we need to completely reduce all overlap relations.

There are 3 overlap relations

$$Ov(x^2, x^2) = 0, Ov(y^2, y^2) = Vxy^2 - Vxyx, \text{ and } Ov(y^2, yx) = -Vxyx + Uxyx.$$

But there are no length 3 paths in \mathcal{N} , hence the three overlap relations must completely reduce to 0 by \mathcal{G} . It follows that $\mathcal{I} = \{0\}$. Thus, $\mathcal{A} = \text{GrAlg}(\mathcal{T})$. Note that for $V = 1$ we have commutative strong Koszul algebras, including $K\{x, y\}/\langle x^2, y^2, yx - xy \rangle$. □

These examples are deceptively easy. In general, $\text{GrAlg}(\mathcal{T})$ is a proper nontrivial variety. The next example gives some indication of the complexity of the varieties.

Example 5.4. Let Q be the quiver with one vertex and 3 loops, x, y , and z . Order them by $z \succ y \succ x$. Let $\mathcal{T} = \{zy, zx, yx\}$. Then $\mathcal{N} = \{x^i y^j z^k \mid i, j, k \geq 0\}$. We note that the commutative polynomial ring $K\{x, y, z\}/\langle zy - yz, zx - xz, yx - xy \rangle$ is a strong Koszul algebra since $Ov(zy, yx) = -yzx + zyx$ completely reduces to 0 by $\mathcal{G} = \{zy - yz, zx - xz, yx - xy\}$.

By the results in Section 4, the strong Koszul algebras having basis \mathcal{N} (the same as the commutative polynomial ring in 3 variables) are the points in $\text{GrAlg}(\mathcal{T})$. The projective resolution of K over these algebras all have the same shape as the resolution of K over the commutative polynomial ring by Theorem 4.6. Thus $\text{GrAlg}(\mathcal{T})$ has some connection with Artin-Schelter regular algebras of dimension 3.

Now $\mathcal{N}_2 = \{z^2, yz, y^2, xz, xy, x^2\}$ and the length-lexicographic order yields

$$z^2 \succ zy \succ zx \succ yz \succ y^2 \succ yx \succ xz \succ xy \succ x^2$$

Thus, $\mathcal{N}_2(zy) = \{yz, y^2, xz, xy, x^2\} = \mathcal{N}_2(zx)$ and $\mathcal{N}_2(yx) = \{xz, xy, x^2\}$. Hence $\mathcal{A} = K^{13}$ and $\text{GrAlg}(\mathcal{T})$ is a variety in \mathcal{A} .

We let $A, B, \dots, H, L, M, N, P, Q$ denote 13 variables. We set

$$\begin{aligned}
g_{zy} &= zy - Ayz - By^2 - Cxz - Dxy - Ex^2 \\
g_{zx} &= zx - Fyz - Gy^2 - Hxz - Kxy - Mx^2 \\
g_{yx} &= yx - Nxz - Pxy - Qx^2.
\end{aligned}$$

Let $\mathcal{G} = \{g_{zy}, g_{zx}, g_{yx}\}$.

There is only one overlap relation we need to study, namely,

$$Ov(zy, yx) = -Ayzx - By^2x - Cxzx - Dxyx - Ex^2 + Nzxz + Pzxy + Qzx^2.$$

We completely reduce this overlap relation and collect the polynomials that are the coefficients of elements of \mathcal{N}_3 . By brute force computation, the ideal of $\text{GrAlg}(\mathcal{T})$ contains 8 polynomials, 2 of total degree 3 and 6 of total degree 4.

Below is the polynomial that is the coefficient of xyz :

$$\begin{aligned}
&-AP - A^2LN - AFMN - BP - BPNA - BQNF - CF + NFP + N^2DA + N^EF \\
&+ NHA + P^2F + PDNA + PENF + PHA + QFP + BLNA + QMNF \\
&+ MGNP + MPNA + MQNF + MHF
\end{aligned}$$

The dimension of $\text{GrAlg}(\mathcal{T})$ is unclear, as is its irreducibility. \square

Remark 5.5. The connection with Artin-Shelter regular algebras holds in all dimensions. More precisely, consider the commutative polynomial in n variables $R = K\{x_1, x_2, \dots, x_n\}/\langle x_jx_i - x_ix_j \mid 1 \leq i < j \leq n \rangle$. Taking $x_n \succ x_{n-1} \succ \dots \succ x_1$, it is easy to check that R is a strong Koszul algebra. Thus, if $\mathcal{T} = \{x_jx_i \mid 1 \leq i < j \leq n\}$ then $R \in \text{GrAlg}(\mathcal{T})$ and, in this case, $\mathcal{N} = \{x_1^{i_1}x_2^{i_2}\dots x_n^{i_n} \mid i_j \geq 0, 1 \leq i \leq n\}$. Thus $\text{GrAlg}(\mathcal{T})$ consists of all the strong Koszul algebras with graded K -bases \mathcal{N} . Again, for each algebra in $\text{GrAlg}(\mathcal{T})$, the simple module K has a projective resolution the same shape as the projective resolution of K over R by Theorem 4.6.

6. SUBVARIETIES

Fix a quiver Q , a length admissible order \succ on the set of paths \mathcal{B} , and a set \mathcal{T} of paths of length 2. As usual, let $\mathcal{N} = \mathcal{B} \setminus \text{tip}(\langle \mathcal{T} \rangle)$ and $D = \sum_{t \in \mathcal{T}} |\mathcal{N}_2(t)|$. As we have seen, $\text{GrAlg}(\mathcal{T})$ is a variety in $\mathcal{A} = K^D$ and the points of $\text{GrAlg}(\mathcal{T})$ correspond to the strong Koszul algebras with a fixed associated monomial algebra $KQ/\langle \mathcal{T} \rangle$. In this section we introduce subvarieties of $\text{GrAlg}(\mathcal{T})$ that have a distinguished subalgebra and which are intersections of $\text{GrAlg}(\mathcal{T})$ with specified affine subspaces.

Let \mathfrak{r} be a subset of $\{(t, n) \mid t \in \mathcal{T} \text{ and } n \in \mathcal{N}_2(t)\}$ and $\psi: \mathfrak{r} \rightarrow K$. We define $\text{GrAlg}_\psi(\mathcal{T})$ to be the set of the strong Koszul algebras $\Lambda = KQ/I$ (with respect to I and \succ) such that the reduced Gröbner basis of I , $\mathcal{G} = \{g_t \mid t \in \mathcal{T}\}$, where $g_t = t - \sum_{n \in \mathcal{N}_2(t)} c_{t,n}n$ and satisfies the restriction that, for each $(t, n) \in \mathfrak{r}$, $c_{t,n} = \psi((t, n))$.

We break \mathfrak{r} into two disjoint sets; namely, Let $\mathfrak{r}^0 = \{(t, n) \in \mathfrak{r} \mid \psi((t, n)) = 0\}$ and $\mathfrak{r}^+ = \mathfrak{r} \setminus \mathfrak{r}^0$. The *distinguished algebra* in $\text{GrAlg}_\psi(\mathcal{T})$ is $\Lambda^* = KQ/I^*$ where I^* is generated by $\{g_t^* \mid t \in \mathcal{T}\}$ where $g_t^* = t - \sum_{(t, n) \in \mathfrak{r}^+} \psi((t, n))n$. Note that if $\mathfrak{r}^+ = \emptyset$, then $g_t^* = g_t$ and $\Lambda^* = KQ/\langle \mathcal{T} \rangle$. In general, Λ^* may or may not be in $\text{GrAlg}_\psi(\mathcal{T})$, depending on whether or not Λ^* is a strong Koszul algebra. Summarizing, $\text{GrAlg}_\psi(\mathcal{T})$ are the strong Koszul algebras in $\text{GrAlg}(\mathcal{T})$ having reduced Gröbner bases $\{t - \sum_{n \in \mathcal{N}_2(\mathcal{T})} c_{t,n}n\}$ with $c_{t,n} = \psi((t, n))$ for all $(t, n) \in \mathfrak{r}$.

Let \mathfrak{A}_ψ be the affine subspace of \mathcal{A} defined by

$$\mathfrak{A}_\psi = \{(x_{t,n}) \in \mathcal{A} \mid x_{t,n} = \psi((t, n)) \text{ if } (t, n) \in \mathfrak{r}\}$$

. We have the following result, whose proof is left to the reader..

Proposition 6.1. *Let \mathcal{T} be a set of paths of length 2 in a quiver Q . If $\mathfrak{r} \subseteq \{(t, n) \mid t \in \mathcal{T}, n \in \mathcal{N}_2(t)\}$ and $\psi: \mathfrak{r} \rightarrow K$, then*

$$\text{GrAlg}_\psi(\mathcal{T}) = \text{GrAlg}(\mathcal{T}) \cap \mathfrak{A}_\psi.$$

Moreover, $\dim_K(\mathfrak{A}_\psi) = D - |\mathfrak{r}|$. □

By the above Proposition, $\text{GrAlg}_\psi(\mathcal{T})$ could be viewed as living in $K^{D-|\mathfrak{r}|}$. In this setting the distinguished algebra is the algebra associated to the point $\mathbf{0} \in K^{D-|\mathfrak{r}|}$. The next result provides another proof that $\text{GrAlg}_\psi(\mathcal{T})$ is an affine variety in affine $(D - |\mathfrak{r}|)$ -space by explicitly describing the ideal of the variety, viewed as a variety in $K^{D-|\mathfrak{r}|}$.

Theorem 6.2. *Keeping the notation above, $\text{GrAlg}_\psi(\mathcal{T})$ is a subvariety of $\text{GrAlg}(\mathcal{T})$. The subvariety $\text{GrAlg}_\psi(\mathcal{T})$ lives in affine space $K^{D'}$, where $D' = (\sum_{t \in \mathcal{T}} |\mathcal{N}_2(t)|) - |\mathfrak{r}|$.*

Proof. The proof follows the proof of Theorem 3.5 after replacing the variables $y_{t,n}$ with the constants $\psi((t, n))$ for $(t, n) \in \mathfrak{r}$. Thus the polynomials in the ideal of the variety are produced by completely reducing the appropriate overlap relations. (See Example 6.3 below.) The dimension of the underlying affine space is clear. □

Example 6.3. Suppose we are interested in strong Koszul algebras in the same variety as the commutative polynomial ring $R = K\{x, y, z\}/\langle zy - yz, zx - xz, yx - xy \rangle$ with $z \succ y \succ x$. We saw in Example 5.4 that $\text{GrAlg}(\mathcal{T})$ where $\mathcal{T} = \{zy, zx, yx\}$ is extremely complicated. We consider the following subvariety.

Let $\mathfrak{r} = \{(zy, yz), (zy, y^2), (zy, xz), (zy, x^2), (zx, yz), (zx, y^2), (zx, xz), (zx, x^2), (yx, xz), (yx, xy)\}$ and set $\psi((zy, yz)) = \psi((zx, xz)) = \psi((yx, xy)) = 1$ with all other values of ψ being 0. Note that the distinguished algebra in $\text{GrAlg}_\psi(\mathcal{T})$ is R , the commutative polynomial ring in 3 variables.. There three (t, n) s not in \mathfrak{r} : $(zy, xy), (zx, xy), (yx, x^2)$. Let $A = c_{zy, xy}, B = c_{zx, xy}$ and $C = c_{yx, x^2}$. We have

$$\mathcal{G} = \{g_{zy} = zy - yz - Axy, g_{zx} = zx - xz + Bxy, g_{yx} = yx - xy + Cx^2\}.$$

It follows that $\text{GrAlg}_\psi(\mathcal{T})$ is the variety whose points (A, B, C) satisfy the property that \mathcal{G} is the reduced Gröbner for the ideal it generates in $K\{x, y, z\}$.

To find ideal of the variety, we must completely reduce every overlap relation . As we noted in Example 5.4, there is only one overlap relation, namely,

$$\text{Ov}(zy, yx) = -yzx - Axyx + zxy + Czx^2.$$

After completely reducing the overlap relation, the polynomials that are coefficients of \mathcal{N}_3 generate \mathcal{I} . The reader can verify that we obtain two polynomials: $BC^2 - AC$ and BC . Thus, $\mathcal{I} = \langle BC^2 - AC, BC \rangle = \langle AC, BC \rangle$. The variety $\text{GrAlg}_\psi(\mathcal{T})$ in K^3 has two irreducible components the plane $\{(A, B, 0)\}$ and the line $\{(0, 0, C)\}$. the commutative polynomial ring lies in the plane $\{(A, B, 0)\}$.

7. FURTHER RESULTS ON STRONG KOSZUL ALGEBRAS

We begin by looking at Λ^{op} , the opposite algebra of Λ . The opposite algebra of Λ is $\{\lambda^{op} \mid \lambda \in \Lambda\}$ with addition and multiplication given by $\lambda^{op} + (\lambda')^{op} = (\lambda + \lambda')^{op}$ and $\lambda^{op} \cdot (\lambda')^{op} = (\lambda' \cdot \lambda)^{op}$. It is well-known that Λ is a Koszul algebra if and only if Λ^{op} is a Koszul algebra. If \succ is a length admissible order, then let \succ^{op} be the order $p^{op} \succ^{op} q^{op}$ if and only if $p \succ q$. In particular, if \succ is the length-(left)-lexicographic order defined in Section 5, then \succ^{op} is the length -(right)-lexicographic order.

Given a quiver Q , define the opposite quiver, Q^{op} in the obvious way. If I is an ideal in KQ , then let $I^{op} = \{x^{op} \mid x \in I\}$. We have the following result, whose proof is left to the reader.

Proposition 7.1. *The algebra $\Lambda = KQ/I$ is a strong Koszul algebra (with respect to I and \succ) if and only if $\Lambda^{op} = KQ^{op}/I^{op}$ is a strong Koszul algebra (with respect to I^{op} and \succ^{op}). \square*

The next result deals with tensoring two strong Koszul algebras. In fact, we prove a result about the reduced Gröbner basis of the tensor of two algebras in general; see Theorem 7.4(1) below. Let $\Lambda = KQ/I$ and $\Lambda' = KQ'/I'$. Define Q^* to be the quiver with vertex set $Q_0 \times Q'_0$ and arrow set $(Q_1 \times Q'_0) \cup (Q_0 \times Q'_1)$, where $(a, w') : (u, w') \rightarrow (v, w')$ if $a : u \rightarrow v$ and $(v, b') : (v, w') \rightarrow (v, x')$ if $b' : w' \rightarrow x'$. If $p = a_1 a_2 \cdots a_r$ and $w' \in Q'_0$, then let (p, w') denote the path $(a_1, w')(a_2, w') \cdots (a_r, w')$. If q' is a path in Q' and $v \in Q_0$, then (v, q') has a similar meaning. If $r = \sum_{p \in \mathcal{B}} \alpha_p p \in KQ$ and $w' \in Q'_0$, then let $(r, w) = \sum_{p \in \mathcal{B}} \alpha_p (p, w')$. Similarly, $(v, \sum_{q'} \beta_{q'} q') = \sum_{q'} \beta_{q'} (v, q')$.

Define $\varphi : KQ^* \rightarrow \Lambda \otimes_K \Lambda'$ as follows. If $(v, w') \in Q_0^*$, $\varphi(v, w') = v \otimes w'$, if $(a, w') \in Q_1 \times Q'_0$, $\varphi(a, w') = a \otimes w'$, and if $(v, b') \in Q_0 \times Q'_1$, $\varphi(v, b') = \otimes b'$. This ring homomorphism is clearly surjective. Let I^* denote the kernel of this morphism. Finally, let \succ and \succ' be length admissible orders on \mathcal{B} , the set of paths in Q , and on \mathcal{B}' , the set of paths in Q' , respectively.

Let \succ^* be the length admissible order on \mathcal{B}^* , the set of paths in Q^* , be defined as follows: On vertices $(u, w') \succ^* (v, x')$ if $w' \succ' x'$ or $w' = x'$ and $u \succ v$. Let p^* be the path $(v_1, q'_1)(p_1, w'_1)(v_2, q'_2) \cdots (p_n, w'_n)$ and \hat{p}^* be the path $(\hat{v}_1, \hat{q}'_1)(\hat{p}_1, \hat{w}'_1)(\hat{v}_2, \hat{q}'_2) \cdots (\hat{p}_m, \hat{w}'_m)$ with $p_i, \hat{p}_i \in \mathcal{B}, q'_j, \hat{q}'_j \in \mathcal{B}'$. Remove all the Q^* vertices from p^* and \hat{p}^* . Then $p^* \succ^* \hat{p}^*$ if $\ell(p^*) > \ell(\hat{p}^*)$ or $\ell(p^*) = \ell(\hat{p}^*)$ and the first arrows from the left in p^* and \hat{p}^* where the arrows differ, we have one of the following 4 cases:

- (1) the arrow in p is (v, b') and the arrow in \hat{p} is (u, b'') with $b', b'' \in Q'_1$ and $b' \succ' b''$.
- (2) the arrow in p is (v, b') and the arrow in \hat{p} is (a, w') with $b' \in Q'_1$ and $a \in Q_1$.
- (3) the arrow in p is (a, w') and the arrow in \hat{p} is (a', w'') with $a, a' \in Q_1$ and $a \succ a'$.
- (4) the arrow in p is (a, w') and the arrow in \hat{p} is (a, w'') with $a \in Q_1$ and $w' \succ' w''$.

The reader may check that \succ^* is a length admissible order.

Before getting to the result on tensors, we let

$$C = \{(u, b')(a, x') - (a, w')(v, b') \mid a : u \rightarrow v \text{ is an arrow in } Q_1, \\ b' : w' \rightarrow x' \text{ is an arrow in } Q'_1\}.$$

We see that $C \subseteq I^*$. Let \mathcal{C} be ideal in $K\mathcal{C}$ generated by C . The elements of \mathcal{C} are called *commutativity relations*. Note that they are quadratic elements in KQ^* . The proof of the following result is left to the reader.

Lemma 7.2. *Let p^* be a path in \mathcal{B}^* of length at least 1. Then there exist paths $p \in \mathcal{B}$, $q' \in \mathcal{B}'$ such that either $\ell(p) \geq 1$ or $\ell(q') \geq 1$ and $p^* - (p, w')(v, q') \in \mathcal{C}$, where $v \in Q_0$ is the end vertex of p and $w' \in Q'_0$ is the start vertex of q' . \square*

We use the following convention: If z is a uniform element in KQ with $uzv = z$, $u, v \in Q_0$ and y' is a uniform element in KQ' with $w'y'x' = y'$, then we define $(z, y') \in KQ^*$ to be $(z, w')(v, y')$. Note that $(z, w')(v, y') - (u, y')(z, x')$ is an element of \mathcal{C} . We also have the following lemma.

Lemma 7.3. *Suppose that $\Lambda = KQ/I$, $\Lambda' = KQ'/I'$ are K -algebras, and KQ^* and I^* are defined above. Let \succ and \succ' be length admissible orders for \mathcal{B} and \mathcal{B}' , respectively. Set $\mathcal{N} = \mathcal{B} \setminus \text{tip}(I)$ and $\mathcal{N}' = \mathcal{B}' \setminus \text{tip}(I')$. Then the set $\{n \otimes_K n' \mid n \in \mathcal{N}, n' \in \mathcal{N}'\}$ is a K -basis for $\Lambda \otimes_K \Lambda'$.*

Proof. By the Fundamental Lemma, we have that \mathcal{N} is a K -basis of Λ and \mathcal{N}' is a K -basis of Λ' . Since we are tensoring over K , the result follows. \square

The first part of the following result is quite general. The proof of the result is somewhat technical but straightforward, and we leave routine checking to the reader.

Theorem 7.4. *Suppose that $\Lambda = KQ/I$, $\Lambda' = KQ'/I'$ are K -algebras, and KQ^* and I^* are defined above. Let \succ and \succ' be length admissible orders for \mathcal{B} and \mathcal{B}' , respectively, and let \succ^* be the length admissible order defined above. Let \mathcal{G} and \mathcal{G}' be the reduced Gröbner bases for I and I' , respectively. Then*

(1)

$$\mathcal{G}^* = \{(g, w') \mid g \in \mathcal{G}, w' \in Q'_0\} \cup \{(v, g') \mid g' \in \mathcal{G}', v \in Q_0\} \cup C$$

is the reduced Gröbner basis for I^ with respect to \succ^* . If \mathcal{G} and \mathcal{G}' are composed of length homogeneous elements, then so is \mathcal{G}^* and in this case \mathcal{G}^* is a graded Gröbner basis for the induced length graded algebra $K\mathcal{G}^*/I^* = \Lambda \otimes_K \Lambda'$.*

(2) *If $\Lambda = KQ/I$ and $\Lambda' = KQ'/I'$ are strong Koszul algebras (with respect to I and \succ) and (with respect to I' and \succ'), respectively, then $\Lambda \otimes_K \Lambda' = KQ^*/I^*$ is a strong Koszul algebra (with respect to I^* and \succ^*),*

Proof. First we show that \mathcal{G}^* generates I^* . Clearly $\mathcal{G}^* \subseteq I^*$. Suppose $X = \sum_i \alpha_{p_i^*} p_i^* \in I^*$, where $\alpha_{p_i^*} \in K$ and $p_i^* \in \mathcal{B}^*$. Then, by Lemma 7.2, since $\mathcal{C} \subset I^*$, by repeatedly applying the commutativity relations, we may assume that $X = \sum_i \alpha_i (p_i, w'_i)(v_i, q'_i)$ where $\alpha_i \in K$, $p_i \in \mathcal{B}$, $v_i \in Q_0$, $q'_i \in \mathcal{B}'$, and $w'_i \in Q'_0$. Next we apply the Fundamental Lemma and write, for each i , $p_i = \iota_i + N_i$ and $q'_i = \iota'_i + N'_i$ where $\iota_i \in I$, $N_i \in \text{Span}_K(\mathcal{N})$ and $\iota'_i \in I'$, $N'_i \in \text{Span}_K(\mathcal{N}')$. Thus,

$$X = \sum_i \alpha_i (\iota_i + N_i, w'_i)(v_i, \iota'_i + N'_i) = \sum_i \alpha_i [(\iota_i, \iota'_i) + (\iota_i, N'_i) + (N_i, \iota'_i) + (N_i, N'_i)].$$

We are assuming that $\varphi(X) = 0$. Since $\iota_i \in I$ and $\iota'_i \in I'$, we see that

$$0 = \varphi(X) = \varphi\left(\sum_i \alpha_i(N_i, N'_i)\right) = \sum_i \alpha_i N_i \otimes N'_i.$$

Hence $0 = \sum_i \alpha_i N_i \otimes N'_i$. Let $N_i = \sum_j \beta_{i,j} n_j$ and $N'_i = \sum_j \beta'_{i,j} n'_j$ with $n_j \in \mathcal{N}$ and $n'_j \in \mathcal{N}'$. Then $\sum_i \alpha_i \beta_{i,j} \beta'_{i,j} = 0$. This implies $\sum_i \alpha_i(N_i, N'_i) = 0$. Thus $X = \sum_i \alpha_i[(\iota_i, \iota'_i) + (\iota, N'_i) + (N_i, \iota'_i)]$ and X is generated by \mathcal{G}^* .

The elements of \mathcal{G}^* are clearly uniform. Overlap relations involving two elements of the form (g, w') with $g \in \mathcal{G}$ completely reduce to 0 using the complete reduction to 0 in KQ by \mathcal{G} . Similarly, the overlap relation of two elements of the form (v, g') with $g' \in \mathcal{G}'$ completely reduce to 0 by \mathcal{G}' . Now suppose $g = t - N \in \mathcal{G}$ with $\text{tip}(g) = t$. Consider $ov((u, b')(a, x'), (t, x'))$ where $(u, b')(a, x') - (a, w')(v, b') \in C$ and $t = a\hat{t}$ (with $a \in Q_1, b \in Q'_1$). Then $ov((u, b')(a, x'), (t, x')) = -(a, w')(v, b')(\hat{t}, x') + (u, b')(N, x')$. Using simple reduction involving elements of C , we commute (u, b') past the elements of \hat{t} and N (changing the vertices as needed) to obtain that $ov((u, b')(a, x'), (t, x'))$ reduces to $-(a, w')(\hat{t}, w')(v, b') + (N, w')(v, b')$. But $-(a, w')(\hat{t}, w')(v, b') + N(v, b') = -(a\hat{t} - N, w')(v, b') = -(g, w')(v, b')$. Since $(g, w) \in \mathcal{G}^*$, $ov((u, b')(a, x'), (t, x'))$ completely reduces to 0. The case of element of the form (v, g') , overlapping a commutativity relation, completely reducing to 0 is similar. This completes the proof that \mathcal{G} is a Gröbner basis for I^* with respect to \succ^* . We leave it to the reader to show that \mathcal{G}^* is the reduced Gröbner basis. This completes the proof of part 1.

The proof of part 2 is straightforward and left to the reader. \square

The next result follows from Proposition 7.1 and Theorem 7.4.

Corollary 7.5. *Let $\Lambda = KQ/I$ be a strong Koszul algebra (with respect to I and \succ). Let \succ^* be the length admissible order defined above where $\Lambda' = \Lambda^{op}$. Then the enveloping algebra $\Lambda \otimes_K \Lambda^{op} = KQ^*/I^*$ is a strong Koszul algebra (with respect to I^* and \succ^*). \square*

8. REMARKS AND QUESTIONS

The goal of this paper was to introduce the variety $\text{GrAlg}(\mathcal{T})$ and its connection to Koszul algebras. We believe this connection will lead to further interesting results, and, to that end, we present a number of open questions. By dropping the restriction that \mathcal{T} is composed of paths of length 2, thus allowing \mathcal{T} to be an arbitrary finite set of paths, one still has a variety $\text{GrAlg}(\mathcal{T})$ whose points correspond to certain graded algebras. In this more general setting, if one drops the length homogeneity restriction, one still obtains an algebraic variety whose points correspond to (not necessarily graded) algebras. Such connections are currently under investigation in [10].

In the case of Koszul algebras that are not strong, one still can look at an appropriate variety with \mathcal{T} being the set $\text{tip}(\mathcal{G})$ where \mathcal{G} is a reduced Gröbner basis with respect to some admissible order. In such a variety, $KQ/\langle \mathcal{T} \rangle$ is not a Koszul algebra since it is a nonquadratic monomial algebra. This leads to the following question:

Question 8.1. Suppose $\Lambda = KQ/I$ is a Koszul algebra such that the reduced Gröbner basis \mathcal{G} is not a quadratic ideal. Then $\Lambda_{Mon} = KQ/\langle \text{tip}(\mathcal{G}) \rangle$ is not a Koszul algebra.

Thus Λ and Λ_{Mon} are in $\text{GrAlg}(\text{tip}(\mathcal{G}))$. Are there necessarily other Koszul algebras in $\text{GrAlg}(\text{tip}(\mathcal{G}))$ and, if so, does the set of Koszul algebras in $\text{GrAlg}(\text{tip}(\mathcal{G}))$ have some geometrical interpretation? In particular, it would be interesting to study $\text{GrAlg}(\text{tip}(\mathcal{G}))$ for the Sklyanin algebras. \square

Question 8.2. If $\Lambda = KQ/I$ is a strong Koszul algebra, is there a length admissible order \succ^\perp on the paths in quiver Q^{op} so that the Koszul dual, $\bigoplus_{n \geq 0} \text{Ext}_\Lambda^n(\overline{\Lambda}, \overline{\Lambda})$, is a strong Koszul algebra, where $\overline{\Lambda}$ is KQ/J with $J = \langle Q_1 \rangle$? We believe the answer is no. \square

One can drop the graded restriction and allow arrows together with paths of length two in a Gröbner basis. The new variety contains $\text{GrAlg}(\mathcal{T})$. It would be interesting to study how these two varieties relate to one another.

Basic geometric questions need to be investigated, some of which are listed below.

Question 8.3. Is $\text{GrAlg}(\mathcal{T})$ irreducible? Given Example 6.3 one expects that the answer, in general, is no. Example 5.2 shows that sometimes $\text{GrAlg}(\mathcal{T})$ is irreducible. What do the irreducible components look like, and is there an interpretation of irreducibility in terms of the algebras? \square

Question 8.4. Does every affine algebraic variety occur as some $\text{GrAlg}(\mathcal{T})$? $\text{GrAlg}_\psi(\mathcal{T})$? \square

Question 8.5. What is the dimension of $\text{GrAlg}(\mathcal{T})$ in terms of some invariant, like \mathcal{T} ? Is there an algorithm to compute it? \square

Question 8.6. Characterize when $\text{GrAlg}(\mathcal{T})$ is a point? \square

Question 8.7. Characterize when $\text{GrAlg}(\mathcal{T})$ is all of affine space? \square

Question 8.8. Suppose $\Lambda = KQ/I$ is a finite dimensional, selfinjective, strong Koszul algebra (with respect to I and \succ), does $\text{GrAlg}(\text{tip}(I))$ necessarily contain other selfinjective, strong Koszul algebras? Does the set of selfinjective strong Koszul algebras have a geometrical interpretation in $\text{GrAlg}(\text{tip}(I))$? \square

Is there a geometric condition on $\text{GrAlg}(\mathcal{T})$ that assures the existence of a finite dimensional selfinjective algebra in $\text{GrAlg} \mathcal{T}$? \square

Question 8.9. Let \mathcal{T} be a set of paths of length 2 in a quiver Q and let \succ be a length admissible order on the set of paths in Q . Let \mathcal{I} be the ideal of the variety $\text{GrAlg}(\mathcal{T})$. Is there a fast way to determine the degrees of a generating set of polynomials in \mathcal{I} or a bound on their degrees? More precisely, overlap relations result in paths of length 3. There is a bound on the number of simple reductions needed to completely reduce to words that are in \mathcal{N}_3 , i.e., nontips of length 3. This number (plus 1) should bound the degree of the polynomials in \mathcal{I} . Is there a fast way to find this number given \mathcal{T} ? \square

REFERENCES

- [1] Anick, David J.; Green, Edward L. *On the homology of quotients of path algebras*. Comm. Algebra 15 (1987), no. 1-2, 309-341.
- [2] Bergman, George M. *The diamond lemma for ring theory*. Adv. in Math. 29 (1978), no. 2, 178-218.
- [3] Conca, Aldo *Koszul algebras and their syzygies*. Combinatorial algebraic geometry, 1-31, Lecture Notes in Math., 2108, Fond. CIME/CIME Found. Subser., Springer, Cham, 2014.
- [4] Föhrberg, R. *Koszul algebras*. Advances in commutative ring theory (Fez, 1997), 337-350, Lecture Notes in Pure and Appl. Math., 205, Dekker, New York, 1999. (16S37)
- [5] Green, E. L.; Happel, D.; Zacharia, D. *Projective resolutions over Artin algebras with zero relations*. Illinois J. Math. 29 (1985), no. 1, 180-190.
- [6] Green, Edward L. *Introduction to Koszul algebras*. Representation theory and algebraic geometry (Waltham, MA, 1995), 45-62, London Math. Soc. Lecture Note Ser., 238, Cambridge Univ. Press, Cambridge, 1997.
- [7] Green, Edward; Huang, Rosa Q. *Projective resolutions of straightening closed algebras generated by minors*. Adv. Math. 110 (1995), no. 2, 314-333.
- [8] Green, Edward L. *Multiplicative bases, Gröbner bases, and right Gröbner bases*. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998). J. Symbolic Comput. 29 (2000), no. 4-5, 601-623.
- [9] Green, Edward L.; Solberg, Øyvind *An algorithmic approach to resolutions*. J. Symbolic Comput. 42 (2007), no. 11-12, 1012-1033.
- [10] Green, Edward L.; Hille, Lutz; Schroll, Sibylle *The geometry of Gröbner bases*, to appear.
- [11] He, Ji-Wei; Van Oystaeyen, Fred; Zhang, Yinhua *Deformations of Koszul Artin-Schelter Gorenstein algebras*. Manuscripta Math. 141 (2013), no. 3-4, 463-483.
- [12] Martínéz-Villa, Roberto; Solberg, Øyvind *Artin-Schelter regular algebras and categories*. J. Pure Appl. Algebra 215 (2011), no. 4, 546-565.
- [13] Smith, S. P. *Point modules over Sklyanin algebras*. Math. Z. 215 (1994), no. 2, 169-177.
- [14] Wilson, George V. *The Cartan map on categories of graded modules*. J. Algebra 85 (1983), no. 2, 390-398.

EDWARD L. GREEN, DEPARTMENT OF MATHEMATICS, VIRGINIA TECH, BLACKSBURG, VA 24061, USA

E-mail address: `green@math.vt.edu`